The Wickelgren Power Law and the Ebbinghaus Savings Function

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Wayne Wickelgren, who died on November 2, 2005 after a long battle with Lou Gerhig's disease, studied the time course of forgetting more assiduously and more effectively than anyone since Hermann Ebbinghaus. In a classic paper, Wickelgren (1974) derived an equation that is remarkable in several respects, including in its ability to characterize the famous Ebbinghaus (1885) savings function. Under typical conditions, Wickelgren's power law reduces to:

$$m = \lambda(1 + \beta t)^{-\psi} \qquad (1)$$

where m is memory strength, and t is time (i.e., the retention interval). The equation has 3 parameters: λ is the state of long-term memory at $t = 0$ (i.e., the degree of learning), ψ is the rate of forgetting, and β is a scaling parameter.

Wixted (2004) showed that Equation 1 provides an accurate description of forgetting curves consisting of data that have been averaged over many subjects. It not only fits the data well in terms of the percentage of variance accounted for—an admittedly weak test—but it also accurately predicts where future points will fall as the retention interval increases, which is a stronger test. An even stronger test would be to accurately predict the course of forgetting using individual subject data because of possible averaging artifacts in group data. However, a practical problem is that such data are usually quite noisy. A rare and notable exception is the savings function reported by Ebbinghaus (1885). Previous work has shown that the Ebbinghaus curve is reasonably well characterized by a 2-parameter power function of the form

$$m = \theta t^{-\psi}, \qquad (2)$$

which can be considered an approximation of Equation 1 (Anderson & Schooler, 1991; Wixted & Ebbesen, 1991). Although Equation 2 offers a much better fit of the savings function than other 2-parameter candidates, it is undefined at $t = 0$, which is theoretically unsatisfying and which limits the equation's practical utility (e.g., it cannot be used to estimate the degree of learning).

The left panel of Figure 1 shows the Ebbinghaus savings function along with the least squares fit of the Wickelgren power law (Equation 1). Not immediately apparent is the fact that there are actually 4 successive fits shown in that graph. In the first, Equation 1 was fit only to the first 5 points (up to 24 hours), but was projected out to 31 days and drawn through all 8 points. In the second, Equation 1 was fit to the first 6 points and then projected out to 31 days. In the third, it was fit to the first 7 points, and in the fourth, it was fit to all 8. Remarkably, the 4 successive fits appear to be a single curve (i.e., they literally fall atop one another). The inset graph shows the fit of the power law using the data from the first 24 hours only. While those 5 points appear to be almost vertically arranged in the larger graph, the general form of forgetting over 24 hours is actually much like the general form of forgetting over 31 days.

The right panel of Figure 1 shows a similar series of fits using another candidate function—the exponential—of the form

$$m = (a - c)e^{-bt} + c \qquad (3)$$

where a is the degree of learning, b is the rate of forgetting, and c is the asymptote (Rubin, Hinton & Wenzel, 1999). The shape of the exponential function often mimics that of the power function (Wixted, 2004), but the two functions differ in one theoretically intriguing respect. Specifically, whereas Equation 1 assumes that the forgetting function descends toward an asymptote of zero, Equation 3 allows for the possibility that it descends toward an asymptote greater than zero. From Figure 1, it is clear that Equation 3 systematically errs by overestimating where the next point will fall as the retention interval increases, such that the estimated asymptote declines as each new retention interval is added (as might be expected if the true asymptote were zero). This result stands in sharp contrast to the fit of the power law, which projects the same course of forgetting for each fit. Thus, the Wickelgren power law and the Ebbinghaus savings function conspire to suggest that forgetting functions ultimately project to an asymptote of zero.

Equation 1 is not only remarkable in its descriptive and predictive accuracy, it also offers a unique practical advantage with regard to characterizing two properties

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of individual subject forgetting functions that have long been of interest to memory researchers, namely, the *degree of learning* and the *rate of forgetting* (cf. White, 1985). Equation 1 has two parameters that correspond to those properties (λ and ψ, respectively). It also includes a scaling parameter (β), which is needed because time is measured in arbitrary units. However, this parameter can be reasonably assumed to remain constant across subjects and across conditions (i.e., subjects can be assumed to scale time in the same way), and doing so greatly reduces the number of parameters that need to be estimated. Specifically, each subject's data in a given condition can be fit by Equation 1, with λ and ψ free to vary across subjects (i.e., the degree of learning and the rate of forgetting are estimated for each subject), but with β constrained to be equal across subjects. For 30 subjects, this would mean estimating 60 parameters (the absolute minimum), plus 1 additional scaling parameter that is common to all subjects, for a total of 61 parameters. By contrast, fitting Equation 3 to 30 individual forgetting functions would require estimating 90 parameters because none of its 3 parameters can be assumed to remain constant across subjects.

Wickelgren's contributions to the study of forgetting go well beyond the equation he proposed (e.g. Wixted 2004). Still, in light of its ability to so accurately characterize the venerable Ebbinghaus savings function, it seems appropriate to recognize the Wickelgren power law as an elegant contribution to the field -- one that was never fully appreciated while he was alive.

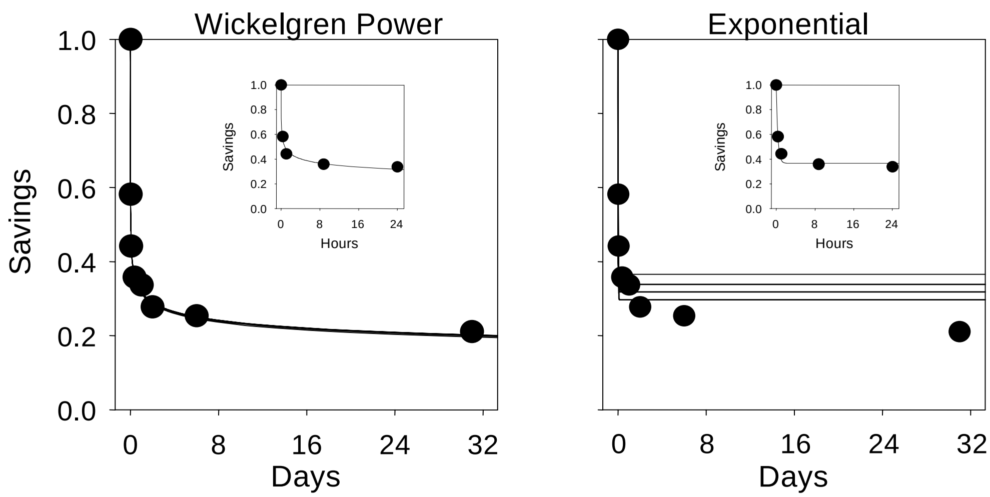

Figure 1. Four successive least-squares fits of Equation 1 (left panel) and Equation 3 (right panel) to the Ebbinghaus (1885) savings curve. The first fit involved the first 5 points, the second involved the first 6, the third involved the first 7, and the fourth involved all 8. Only a single curve is visually apparent in the left panel because the 4 curves fall atop one another. The inset graphs show the fits using only the first 5 points (through 24 hours). For these fits, both the power law and the exponential account for 99% of the variance.

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