BOOK REVIEW

Travels in Poisson Space


Reviewed by Stephen W. Link

Jan Grandell is Biträdande Professor of Mathematical Statistics at the Royal Institute of Technology, Stockholm, Sweden. He received his doctorate in mathematical statistics at the University of Stockholm in 1976. Grandell's work emphasizes the application and analysis of stochastic theory with respect to compound processes and the mathematical theory of risk. Professor Grandell's extensive publications in the field of stochastic processes include the books *Doubly Stochastic Poisson Processes* (1976) and *Aspects of Risk Theory* (1991).


*Mixed Poisson Processes* consolidates recent advances in Poisson theory. Most advances occur later than Haight's (1967) *Handbook of the Poisson Distribution*, but Grandell begins at an even earlier stage. Blending together history and precision, the author weaves a story of Poisson theory that is both stimulating and challenging—well worth a graduate-level course of study into this fundamental...
basis for discrete processes. The treatment is sufficiently broad to be of interest to physicists, biologists, psychologists, probabilists, and financial analysts.

The Poisson distribution (Poisson, 1837) is familiar to psychophysicists who use it as a foundation for models of nervous integration (Rashevsky, 1937, 1960), signal detection (Creelman, 1961), probability learning (Suppes & Donio, 1967), sums of neural events, (McGill & Gibbon, 1965; Link, 1992), and counting processes (Townsend and Ashby, 1983). Grandell credits Ove Lundberg with introducing and developing many seminal discoveries in a thesis entitled On Random Processes and Their Application to Sickness and Accident Statistics (1940, 1964). Lundberg, who acknowledged a debt to William Feller, introduced the idea of compound Poisson processes, what we today would define as weighted or mixed Poisson processes. Significantly, both Lundberg and Feller began their outstanding contributions to the advancement of Poisson theory as part of Harold Cramer’s group in Stockholm.

Grandell begins this treatise by considering the Poisson distribution with parameter \( \lambda > 0 \), Po(\( \lambda \)). The distribution defines the probability that a number of events, \( N \), will occur within a unit of time, such as the number of nonfatal accidents that an individual may experience during a year. The parameter \( \lambda \) defines the accident proneness of an individual and may also depend upon situational factors.

Thus, the Poisson distribution is conditioned on \( \lambda \) but the parameter \( \lambda \) may vary from one person to another. Considering the marginal distribution of \( N \) gives rise to a mixed Poisson distribution that yields a complete description of the number of accidents incurred by a population of individuals during one year. In particular, if \( A \) is the random variable (RV) for \( \lambda \), and \( U \) (known as the structure distribution) its distribution, then the distribution of the number of nonfatal accidents, \( N \), is determined by

\[
P[N = k] = \int_0^\infty \frac{\lambda^k}{k!} e^{-\lambda} du(\lambda), \quad k = 0, 1, \ldots
\]

Although the probabilistic phenomena are Poisson distributed for each individual, the Poisson characteristics disappear from the distribution of \( N \). Quite clearly \( E(N) = E(A) \) and \( Var(N) = E(A) + Var(A) \). If \( Var(A) = 0 \) then \( Var(N) = E(N) \), the defining feature of a Poisson distributed RV. The mixed Poisson distribution shows greater variability than the Poisson. These are the ideas that began the application of the Poisson distribution to a variety of practical problems described by Greenwood and Yule (1920) and Newbold (1926).

The Poisson process follows a similar treatment but now the number of events that occur is defined with respect to time. The parameter of the Poisson process is \( \lambda t \). Continuing the example of accidents, imagine an individual insured against traffic accidents. The random variable \( A \) will follow some probability distribution \( U \) defining the accident proneness of each member of the insured population. This creates the mixed Poisson process where \( N(t) \) is the number of accidents occurring by time \( t \). Now, in contrast to the mixed Poisson distribution,

\[
P[N(t) = k] = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} du(\lambda), \quad k = 0, 1, \ldots
\]
At time 0, \( N(0) = 0 \), but by time \( t \), there are \( N(t) \) accidents. Given this information about each individual the insurer wants to predict the number of accidents during the next premium period of duration \( h \), that is, \( N(t + h) - N(t) \). The desired expected number of accidents, conditioned on \( N(t) \), is easily determined by application of the principles of conditional expectation:

\[
E[N(t + h) - N(t) | N(t) = n] = E[E[N(t + h) - N(t) | A, N(t) = n] N(t) = n]
= hE[A | N(t) = n].
\]  

(3)

If \( A \) has expected value \( \bar{\lambda} \) and is independent of \( N(t) \) then the required result is \( h\bar{\lambda} \), a sensible number.

To summarize, the mixed Poisson process can be defined by principles suggested by Newbold (1926).

1. Individuals differ from each other with respect to accident proneness
2. No contagion, accidents of the past do not influence the future
3. Individual accident proneness remains constant across time (stationarity).

So, what’s to study? In contrast to the Newbold principles, let us suppose that all individuals have the same accident proneness, more accidents are likely to increase the probability of future accidents, and accident proneness depends on time. All these assumptions militate against the Newbold principles, indeed, defy them. Yet these assumptions and the Newbold principles give rise to the same probability distribution for the number of accidents. This surprising outcome suggests the existence of deeper stochastic properties that warrant intensive investigation.

Mixed Poisson processes is self-contained and divided into convenient chapters that introduce the elementary players: Poisson distributions and processes, and then Cox, Gauss–Poisson, and mixed renewal processes. Having developed a background and vocabulary Grandell examines mixed Poisson processes within the confines of birth, stationary point, and general point processes. Applications to reliability theory, especially useful for actuaries and light bulb manufacturers, follow. The applications to mental events are direct.

Many stochastic theorists may direct their attention toward Chapter 8, “Compound mixed Poisson distributions.” These distributions arise from considerations of random sums of discrete RVs. In particular, if \( N \) is a discrete RV and \( \{Z_k\}^\infty_{k=1} \) is a sequence of non-negative independent and identically distributed (IID) RVs, then the RV

\[
Y = \sum_{k=1}^{N} Z_k
\]  

(4)

is said to be compound distributed or a random sum. When \( N \) is Poisson distributed \( Y \) is said to have a compound Poisson distribution.

A straightforward analysis of \( Y \) establishes that

\[
E[Y] = E[E[Y | N]] = \mu E[N],
\]  

(5)
where $\mu$ is the expected value of $Z_k$. The result is also obtainable as an “approximation” through application of Wald's (1947) identity to bounded random sums of IIDRVs. As for the variance,

$$\text{Var}[ Y ] = E[\text{Var}[ Y | N ]] + \text{Var}[ E[ Y | N ]]$$

$$= \sigma^2 E[ N ] + \mu^2 \text{Var}[ N ],$$

(6)

where $\sigma^2$ is the variance of $Z_k$.

Much interest might be focused on such random sums, especially when $N$ is the consequence of a more basic underlying process. Establishing bounds on the probability of exceeding a critical value of $Y$ is especially important for insurance companies, but restrictions on the number of claims, $N(t)$ are also useful. This is Grandell's forte, having established in 1970 a simple and fundamental result on the probability that $N(t)$ exceeds $n$. Perhaps the detail and length (63 pages) of this longest chapter is a measure of Grandell's interest.

The book's last chapter, entitled “The risk business,” distinguishes between mixed Poisson and Ammeter (1948) processes. These models analyze the inputs of receipts and claims to establish ruin probabilities in an effort to avert catastrophic failure (think Lloyd's of London). In some cases I believe the results are also easily established by direct application of Wald's Identity.

De Finetti, in his two-volume tour de force Theory of probability (1974) proclaimed that objective probability does not exist. Quite a headache for the renowned Professor of Probability at the University of Rome, but an even more disturbing possibility for author Jan Grandell. If objective probability does not exist the grand legacy of Ove Lundberg's genius may be of questionable value. But, let's not beat too hasty a retreat from our probabilistic analyses. Instead, let this book take us on a terrific ride through the space of Poisson theory, motivated by practical problems, and devoted to deep analysis of unsuspected equivalences in seemingly disparate models.

REFERENCES


